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Topology of the space of smooth solutions to the Liouville equation

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Abstract

We prove that the space of smooth initial data and the space of smooth solutions to the Liouville equation are homeomorphic.

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1. Introduction

The purpose of this paper is to prove the theorem that the space $\mathcal{M}$ of smooth, $C^\infty(\mathbb{R}^2)$, solutions to the Liouville equation [1]:

$$(\partial_t^2 - \partial_x^2)F(t,x) + \frac{m^2}{2}\exp F(t,x) = 0, \quad m > 0$$

with the topology of almost uniform convergence with all derivatives is *homeomorphic* to the space $C^\infty(\mathbb{R}) \times C^\infty(\mathbb{R})$ of smooth initial data.

The existence of such homeomorphism guarantees the stability of solutions of the Liouville equation with respect to the perturbation of initial data, i.e., if a series of smooth initial data converges almost uniformly with all its derivatives to $(f_1, f_2) \in C^\infty(\mathbb{R}) \times C^\infty(\mathbb{R})$, then the series of corresponding solutions to the Liouville equation converges almost uniformly with all its derivatives to the solution corresponding to the initial data (f_1, f_2).

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The proof of the theorem consists of two parts which are presented in Sections 2 and 3. In Section 2 we quote some results of Jorjadze et al. [2] concerning the solution of the Cauchy problem for the Liouville equation and we prove these results. We present the construction of the smooth solution F to the Liouville equation for given initial data. For given $f_1, f_2 \in C^\infty(\mathbb{R})$ we introduce the functions $u, w \in C^\infty(\mathbb{R})$ defined by (11) and (12). Then, we consider the bases of solutions to the linear equations: $g_i'' = ug_i$ $(i = 2, 4)$, Eq. (13), and $g_j'' = wg_j$ $(j = 1, 3)$, Eq. (14). We use g_1, g_2, g_3 and g_4 to define the map $G \in C^\infty(\mathbb{R}^2)$, Eq. (10). Finally, we get the solution to the Liouville equation in the form $F := -\log[(m^2/16)G^2]$, Eq. (15). The main result of Section 2 is Proposition 1 giving explicitly the relation between $f_1, f_2 \in C^\infty(\mathbb{R})$ and $F \in \mathcal{M}$. In Section 3 we introduce a few mappings between some subspaces of $C^\infty(\mathbb{R}^M, \mathbb{R}^N)$, e.g., the function u given by (11) defines the mapping (30) $\mathcal{A} : C^\infty(\mathbb{R})^2 \ni (f_1, f_2) \to \mathcal{A}(f_1, f_2) := u \in C^\infty(\mathbb{R})$. Similarly, we define another mappings: $\mathcal{B}, \mathcal{C}, \ldots, \mathcal{I}$ given by (31), (32), ..., (38). Next we prove that each of these mappings is continuous. The bijection $\mathcal{I} : C^\infty(\mathbb{R})^2 \ni (f_1, f_2) \to \mathcal{I}(f_1, f_2) = F \in \mathcal{M}$ is continuous as the composition of continuous maps. The continuity of the inverse mapping $\mathcal{I}^{-1} : \mathcal{M} \to C^\infty(\mathbb{R})^2$ is immediate. This completes the proof.

Let us introduce the notation and let us recall some results concerning smooth mappings to make the paper self-contained:

From now on, $\mathbb{N} := \{0, 1, \ldots\}$ and $\mathbb{N}^\times := \mathbb{N} \setminus \{0\}$. In $\mathbb{R}^N$, where $N \in \mathbb{N}^\times$, we use the norm

$$\|\cdot\| : \mathbb{R}^N \ni y \to \| y \| := \left(\sum_{j=1}^{N} (y^j)^2 \right)^{1/2} \in \mathbb{R}.$$

$\mathbb{R}^M = \bigcup_{\alpha=1}^{\infty} K_\alpha^M$, where $K_\alpha := [-\alpha, \alpha]$ and $\alpha, M \in \mathbb{N}^\times$; $\partial^\beta := \partial_1^{\beta_1} \cdot \partial_2^{\beta_2} \cdots \partial_M^{\beta_M}$, $\beta := (\beta_1, \beta_2, \ldots, \beta_M)$, where $\beta_k \in \mathbb{N}$ $(k = 1, 2, \ldots, M)$, $|\beta| := \sum_{j=1}^{M} \beta_j$. $C^\infty(\mathbb{R}^M, \mathbb{R}^N)$, where $M, N \in \mathbb{N}^\times$, denotes the space of all smooth mappings $\mathbb{R}^M \to \mathbb{R}^N$ with the topology of almost uniform convergence with all derivatives, i.e., the topology of $C^\infty(\mathbb{R}^M, \mathbb{R}^N)$ is defined by the family of seminorms $p_{\alpha\beta}^{MN}$, where

$$p_{\alpha\beta}^{MN} : C^\infty(\mathbb{R}^M, \mathbb{R}^N) \ni f \longrightarrow p_{\alpha\beta}^{MN}(f) := \sup_{x \in K_\alpha^M} \| (\partial^\beta f)(x) \| \in \mathbb{R},$$

and where $\beta \in \mathbb{N}^M, \alpha \in N^\times$.

We say that the sequence $(f_n)_{n=0}^{\infty}$ of elements of $C^\infty(\mathbb{R}^M, \mathbb{R}^N)$ converges to $f \in C^\infty(\mathbb{R}^M, \mathbb{R}^N)$ iff $\forall (\alpha, \beta) \in \mathbb{N}^\times \times \mathbb{N}^M : \lim_{n\to\infty} p_{\alpha\beta}^{MN}(f - f_n) = 0$. The space $C^\infty(\mathbb{R}^M, \mathbb{R}^N)$ is a complete topological vector space, which in particular means that the operation of multiplication by a scalar

$$\cdot : \mathbb{R} \times C^\infty(\mathbb{R}^M, \mathbb{R}^N) \ni (\lambda, f) \to \lambda \cdot f \in C^\infty(\mathbb{R}^M, \mathbb{R}^N),$$

and the operation of vector addition

$$+ : C^\infty(\mathbb{R}^M, \mathbb{R}^N) \times C^\infty(\mathbb{R}^M, \mathbb{R}^N) \ni (f, g) \to f + g \in C^\infty(\mathbb{R}^M, \mathbb{R}^N)$$

are continuous. $C^\infty(\mathbb{R}^M, \mathbb{R}^N)$ is an example of a Fréchet space.

In what follows $C^\infty(\mathbb{R}^M) := C^\infty(\mathbb{R}^M, \mathbb{R})$; small latin letters denote maps $\mathbb{R} \to \mathbb{R}$, i.e., $f, g, h, \ldots \in C^\infty(\mathbb{R})$; $\mathbb{R}^2 \to \mathbb{R}$ maps are denoted by capital latin letters, i.e., $F, G, H, \ldots \in C^\infty(\mathbb{R}^2)$; $\mathbb{R} \to \mathbb{R}^2$ maps are denoted by capital greek letters, i.e., $\Phi, \Psi, \Omega, \ldots \in C^\infty(\mathbb{R}, \mathbb{R}^2)$.

We denote $p_{\alpha\beta} := p^{11}_{\alpha\beta}$, $r_{\alpha\beta} := p^{12}_{\alpha\beta}$, $q_{\alpha\beta} := p^{21}_{\alpha\beta}$ for $\beta = (\beta_1, \beta_2)$; $\partial_t = \partial_1$, $\partial_x = \partial_2$, $\Box := \partial_t^2 - \partial_x^2$, for $(t, x) \in \mathbb{R}^2$.

There are two subspaces of $C^\infty(\mathbb{R}^2)$ which are of primary importance in our considerations:

$$C^\infty_+(\mathbb{R}^2) := \{F \in C^\infty(\mathbb{R}^2) : F(\mathbb{R}^2) \subseteq]0, \infty[\}$$

and

$$\mathcal{M} := \{F \in C^\infty(\mathbb{R}^2) : \Box F + (m^2/2) \exp F = 0\},$$

where $m > 0$ is a fixed real number.

The topologies on $C^\infty_+(\mathbb{R}^2)$ and $\mathcal{M}$ are induced topologies from $C^\infty(\mathbb{R}^2)$. It means that a sequence $(F_n)_{n=0}^\infty$ of elements from $C^\infty_+(\mathbb{R}^2)$ (or $\mathcal{M}$) converges to $F \in C^\infty_+(\mathbb{R}^2)$ (or $F \in \mathcal{M}$), iff it converges to F in $C^\infty(\mathbb{R}^2)$.

In the space of all linear mappings $\mathbb{R}^2 \to \mathbb{R}^2$ we use the norm

$$\|\cdot\| : B(\mathbb{R}^2) \ni A \to \| A \| := \sup_{\|x\|=1} \| A(x) \| \in \mathbb{R}.$$

2. Smooth solutions of the Liouville equation

We examine properties of a smooth solution to the Liouville equation, i.e., of class $C^\infty(\mathcal{O})$, where $\mathbb{R}^2 \supseteq \mathcal{O}$ is an open subset different to $\emptyset$. However, all proofs can be easily modified to include the solutions of class $C^k(\mathbb{R}^2)$, for $k \geq 2$.

Lemma 1. *Let* $g_i \in C^\infty(\mathbb{R})$ $(i = 1, 2, 3, 4)$ *are such that*

$$g_1 g_3' - g_1' g_3 = 1, \quad g_2 g_4' - g_2' g_4 = -1 \tag{1}$$

and let

$$G : \mathbb{R}^2 \ni (t, x) \to G(t, x) := g_1(x + t) g_2(x - t) + g_3(x + t) g_4(x - t) \in \mathbb{R}.$$

Then we have

$$(G^{-1}(0) \neq \emptyset) \Longrightarrow (G^{-1}(0) \cap \{(0, x) \in \mathbb{R}^2 : x \in \mathbb{R}\} \neq \emptyset).$$

Proof. Let $\xi := x + t, \quad \eta := x - t$. Making use of

$$(G(t_0, x_0) = 0) \Longleftrightarrow (g_1(\xi_0) g_2(\eta_0) = -g_3(\xi_0) g_4(\eta_0)) \tag{2}$$

we see that the condition

$$G(t_0, x_0) = 0, \qquad (\partial_t G)(t_0, x_0) = (\partial_x G)(t_0, x_0)$$

leads to

$$g_1(\xi_0)g_2'(\eta_0) + g_3(\xi_0)g_4'(\eta_0) = 0.$$

Multiplying this equation by $g_2(\eta_0)$ and using (1) gives $g_3(\xi_0) = 0$. Similarly, multiplying by $g_4(\eta_0)$ leads to $g_1(\xi_0) = 0$. Thus, we have

$$\begin{pmatrix} G(t_0, x_0) = 0 \\ (\partial_t G)(t_0, x_0) = (\partial_x G)(t_0, x_0) \end{pmatrix} \Longrightarrow (g_1(t_0, x_0) = 0 = g_3(t_0, x_0)),$$

contrary to (1). In the same manner we can see that

$$\begin{pmatrix} G(t_0, x_0) = 0 \\ (\partial_t G)(t_0, x_0) = -(\partial_x G)(t_0, x_0) \end{pmatrix} \Longrightarrow (g_2(t_0, x_0) = 0 = g_4(t_0, x_0)),$$

which again contradicts (1). Therefore, we have

$$((t_0, x_0) \in G^{-1}(0)) \Longrightarrow ((\partial_t G)(t_0, x_0) \neq \pm(\partial_x G)(t_0, x_0)). \tag{3}$$

The condition $(\partial_x G)(t_0, x_0) = 0$ means that

$$0 = g_1'(\xi_0)g_2(\eta_0) + g_1(\xi_0)g_2'(\eta_0) + g_3'(\xi_0)g_4(\eta_0) + g_3(\xi_0)g_4'(\eta_0). \tag{4}$$

Multiplying (4) by $g_1(\xi_0)g_4(\eta_0)$ and using (2) gives

$$g_1(\xi_0)^2 + g_4(\eta_0)^2 = 0. \tag{5}$$

Similarly, multiplying (4) by $g_3(\xi_0)g_2(\eta_0)$ and using (2) leads to

$$g_3(\xi_0)^2 + g_2(\eta_0)^2 = 0. \tag{6}$$

Since (5) and (6) contradict (1), we conclude that

$$((t_0, x_0) \in G^{-1}(0)) \Longrightarrow ((\partial_x G)(t_0, x_0) \neq 0), \tag{7}$$

which means that either $G^{-1}(0) = \emptyset$ or $G^{-1}(0)$ is a one-dimensional C^∞ submanifold of $\mathbb{R}^2$. Suppose that $G^{-1}(0) \neq \emptyset$ and let us denote by M the connected component of $G^{-1}(0)$. By (3) we have that M cannot be a compact subset of $\mathbb{R}^2$. Since M is closed in $\mathbb{R}^2$, it cannot be bounded in $\mathbb{R}^2$. From (7) we conclude that M can be parametrized by $t \in \mathbb{R}$. Let $\pi : \mathbb{R}^2 \ni (t, x) \to \pi(t, x) := t \in \mathbb{R}$. Since M is closed, $\pi(M) \subseteq \mathbb{R}$ is closed in $\mathbb{R}$. Hence $\emptyset \neq \pi(M) \subseteq \mathbb{R}$ is both closed and open (homeomorphic to $\mathbb{R}$). Therefore $\pi(M) = \mathbb{R}$. Finally, we obtain

$$G^{-1}(0) \cap \{(0, x) \in \mathbb{R}^2 : x \in \mathbb{R}\} \neq \emptyset. \qquad \square$$

By [1] we get the following lemma.

Lemma 2. *Let* $\mathbb{R}^2 \supseteq \mathcal{O}$ *be an open subset and let*

$$\mathcal{O}_1 := \{(x + t) \in \mathbb{R} : (t, x) \in \mathcal{O}\}, \qquad \mathcal{O}_2 := \{(x - t) \in \mathbb{R} : (t, x) \in \mathcal{O}\}.$$

Suppose $F \in C^\infty(\mathcal{O})$*, then the following are equivalent:*
1. $\Box F = -(m^2/2) \exp F$.

2. *There exist* $g_1, g_3 \in C^\infty(\mathcal{O}_1)$ *and* $g_2, g_4 \in C^\infty(\mathcal{O}_2)$ *satisfying* $g_1 g_3' - g_1' g_3 = 1$ *and* $g_2 g_4' - g_2' g_4 = -1$ *such that* $F : \mathcal{O} \ni (t, x) \to F(t, x) := -\log(m^2/16)[g_1(x + t)g_2(x - t) + g_3(x + t)g_4(x - t)]^2 \in \mathbb{R}$.

Lemmas 1 and 2 lead to the following corollary.

Corollary 1. *If* $\mathbb{R}^2 \supseteq \mathcal{O}$ *is an open set such that* $\{(0, x) \in \mathbb{R}^2 : x \in \mathbb{R}\} \subseteq \mathcal{O}$ *and* $F \in C^\infty(\mathcal{O})$ *satisfies the Liouville equation* $\Box F = -(m^2/2)\exp F$, *then there exists* $\tilde{F} \in C^\infty(\mathbb{R}^2)$ *such that* $\Box \tilde{F} = -(m^2/2)\exp \tilde{F}$ *and* $\tilde{F}_{|\mathcal{O}} = F$.

Proposition 1. *Suppose* $f_1, f_2, g_1, g_2, g_3, g_4 \in C^\infty(\mathbb{R})$ *and let*

$$g_1 g_3' - g_1' g_3 = 1, \tag{8}$$

$$g_2 g_4' - g_2' g_4 = -1, \tag{9}$$

$$G : \mathbb{R}^2 \ni (t, x) \to G(t, x) := g_1(x + t)g_2(x - t) + g_3(x + t)g_4(x - t) \in \mathbb{R}, \tag{10}$$

$$u := \frac{1}{16}[(f_1' - f_2)^2 - 4(f_1' - f_2)' + m^2 \exp f_1], \tag{11}$$

$$w := \frac{1}{16}[(f_1' + f_2)^2 - 4(f_1' + f_2)' + m^2 \exp f_1]. \tag{12}$$

Then

1. *If* $g_1, \ldots, g_4$ *satisfy the equations*

$$g_i'' = u g_i \quad \text{for } i = 2, 4 \tag{13}$$

and

$$g_j'' = w g_j \quad \text{for } j = 1, 3, \tag{14}$$

then the map

$$F : \mathbb{R}^2 \ni (t, x) \to F(t, x) := -\log \frac{m^2}{16} G^2(t, x) \in \mathbb{R} \tag{15}$$

is a solution of the Liouville equation with the initial data

$$F(0, \cdot) = f_1, \qquad (\partial_t F)(0, \cdot) = f_2. \tag{16}$$

2. *A solution of the Liouville equation satisfying* (16) *is given by* (15), *where* $g_1, \ldots, g_4$ *satisfy* (13) *and* (14).

Proof.

(1) Suppose g_1, g_3 and $\tilde{g}_1, \tilde{g}_3$ satisfy (8) and (14). Hence there exists

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$$

such that $g_1 = a\tilde{g}_1 + b\tilde{g}_3$ and $g_3 = c\tilde{g}_1 + d\tilde{g}_3$. Such an exchange of functions corresponds to the Bianchi transformation [1] and does not change the form of solution (15). If g_2 and g_4 satisfy (9) and (13), then the functions [1]

$$g_1 := -\frac{4}{m}\exp\left(-\frac{1}{2}f_1\right)\left[g_4' + \frac{1}{4}(f_1' - f_2)g_4\right], \tag{17}$$

$$g_3 := \frac{4}{m}\exp\left(-\frac{1}{2}f_1\right)\left[g_2' + \frac{1}{4}(f_1' - f_2)g_2\right] \tag{18}$$

satisfy (8) and (14). In what follows we assume that the general form of g_1 and g_3 are given by (17) and (18). One can easily check that for G defined by (10) we have

$$\forall x \in \mathbb{R} : G(0, x) = \frac{4}{m}\exp\left(-\frac{1}{2}f_1(x)\right) > 0. \tag{19}$$

By Lemma 1 we have that F given by (15) is well defined on $\mathbb{R}^2$ ($F \in C^\infty(\mathbb{R}^2)$) and satisfies (16).

(2) We have shown that there exists $F \in C^\infty(\mathbb{R}^2)$ satisfying (16). By Lemma 2, F is of the form (15) for $g_1, \ldots, g_4 \in C^\infty(\mathbb{R})$ satisfying (8) and (9). Now, we shall show (see [1]) that $g_1, \ldots, g_4$ can be a solution of (13) and (14) with u and w given by (11) and (12). We define

$$\aleph : \mathbb{R} \ni x \to \aleph(x) := G(0, x) \in \mathbb{R}$$

and

$$\hbar : \mathbb{R} \ni x \to \hbar(x) := g_1'(x)g_2'(x) + g_3'(x)g_4'(x) \in \mathbb{R}.$$

One has $\aleph'/\aleph = -\frac{1}{2}f_1'$ and

$$f_2 = (\partial_t F)(0, \cdot) = \frac{2}{\aleph}(\aleph' - 2(g_1'g_2 + g_3'g_4)) = -f_1' - \frac{4}{\aleph}(g_1'g_2 + g_3'g_4)$$
$$= \frac{2}{\aleph}(2(g_1g_2' + g_3'g_4) - \aleph') = f_1' + \frac{4}{\aleph}(g_1g_2' + g_3g_4')$$

which leads to

$$\tfrac{1}{4}(f_1' + f_2)\aleph = -g_1'g_2 - g_3'g_4, \tag{20}$$

$$\tfrac{1}{4}(f_1' - f_2)\aleph = g_1g_2' + g_3g_4'. \tag{21}$$

Suppose now that g_1 and g_3 satisfy (13), and g_2 and g_4 satisfy (14) for some $u, w \in C^\infty(\mathbb{R})$. Taking derivative of (20) yields

$$\tfrac{1}{4}(f_2 + f_1')'\aleph + \tfrac{1}{4}(f_2 + f_1')\aleph' = -w\aleph - \hbar. \tag{22}$$

By analogy, we get from (21):

$$\tfrac{1}{4}(f_2 - f_1')'\,\aleph + \tfrac{1}{4}(f_2 - f_1')\aleph' = u\aleph + \hbar. \tag{23}$$

By direct calculations, we get

$$(\partial_t^2 F)(0, \cdot) = \frac{1}{2}f_2^2 + 4\frac{\hbar}{\aleph} - 2(u + w) \tag{24}$$

Since $(\partial_x^2 F)(0,\cdot) = f_1''$, we get

$$f_1'' = 2\left(\frac{\aleph'}{\aleph}\right)^2 - 2\frac{\aleph''}{\aleph} \tag{25}$$

The map F satisfies the Liouville equation $\Box F = -(m^2/2)\exp F$ on $\mathbb{R}^2$. By (24) and (25), for $(0,x) \in \mathbb{R}^2$ we get

$$\frac{1}{2}f_2^2 + 4\frac{\hbar}{\aleph} - 2(u+w) + 2\frac{\aleph''}{\aleph} - 2\left(\frac{\aleph'}{\aleph}\right)^2 = -\frac{m^2}{2}\exp f_1. \tag{26}$$

Since

$$2\aleph''/\aleph = 2(u+w) + 4\hbar/\aleph \quad \text{and} \quad (\aleph'/\aleph)^2 = (1/4)(f_1')^2$$

Eq. (26) leads to

$$\frac{\hbar}{\aleph} = -\frac{1}{16}[(f_2 - f_1')(f_2 + f_1') - m^2 \exp f_1]. \tag{27}$$

By (27) and (22) we get

$$w = \frac{1}{16}(f_1' - f_2)^2 - \frac{1}{4}(f_1' - f_2)' + \frac{m^2}{16}\exp f_1. \tag{28}$$

Similarly, (27) and (23) give

$$u = \frac{1}{16}(f_1' + f_2)^2 - \frac{1}{4}(f_1' + f_2)' + \frac{m^2}{16}\exp f_1. \qquad \Box \tag{29}$$

3. Homeomorphism of the space of initial data and the space of solutions

We define the following mappings:

$$\mathcal{A} : C^\infty(\mathbb{R})^2 \ni (f_1, f_2) \longrightarrow \mathcal{A}(f_1, f_2) := u \in C^\infty(\mathbb{R}), \tag{30}$$

where u is given by (11).

$$\mathcal{B} : C^\infty(\mathbb{R}) \ni u \longrightarrow \mathcal{B}(u) := (g_2, g_4) \in C^\infty(\mathbb{R})^2, \tag{31}$$

where g_2 and g_4 are defined by

$$g_2'' = ug_2,\ g_2(0) = 0,\ g_2'(0) = 1 \quad \text{and} \quad g_4'' = ug_4,\ g_4(0) = 1,\ g_4'(0) = 0.$$

Remark 1. *Maps g_2 and g_4 satisfy* (9).

$$\mathcal{C} : C^\infty(\mathbb{R})^4 \ni (f_1, f_2, g_2, g_4) \longrightarrow \mathcal{C}(f_1, f_2, g_2, g_4) := (g_1, g_3) \in C^\infty(\mathbb{R})^2, \tag{32}$$

where g_1 *and* g_3 *are given by* (17) *and* (18).

$$\mathcal{D} : C^\infty(\mathbb{R})^2 \ni (f_1, f_2) \longrightarrow \mathcal{D}(f_1, f_2) := (f_1, f_2, f_1, f_2) \in C^\infty(\mathbb{R})^4, \tag{33}$$

$$\mathcal{E} : C^\infty(\mathbb{R}^2) \ni G \longrightarrow \mathcal{E}(G) := \frac{m^2}{16} G^2 \in C^\infty(\mathbb{R}^2), \tag{34}$$

$$\mathcal{F} : C^\infty_+(\mathbb{R}^2) \ni H \longrightarrow \mathcal{F}(H) := -\log H \in C^\infty(\mathbb{R}^2), \tag{35}$$

$$\mathcal{G} : C^\infty(\mathbb{R})^4 \ni (g_1, g_3, g_2, g_4) \longrightarrow \mathcal{G}(g_1, g_3, g_2, g_4) := G \in C^\infty(\mathbb{R}^2), \tag{36}$$

where G *is defined by* (10).

$$\mathcal{H} := \mathcal{E} \circ \mathcal{G} \circ (\mathcal{C} \times id_2) \circ (id_2 \times \mathcal{D}) \circ (id_2 \times \mathcal{B}) \circ (id_2 \times \mathcal{A}) \circ \mathcal{D}, \tag{37}$$

where $id_2 := id_{C^\infty(\mathbb{R})^2}$.

Remark 2.

$$\mathcal{H} : C^\infty(\mathbb{R})^2 \ni (f_1, f_2) \longrightarrow \mathcal{H}(f_1, f_2) := \frac{m^2}{2} G^2 \in C^\infty(\mathbb{R}^2).$$

By Lemma 1 *we have the following corollary.*

Corollary 2.

$$\mathcal{H}(C^\infty(\mathbb{R})^2) \subseteq C^\infty_+(\mathbb{R}^2)$$

Let

$$\mathcal{I} := \mathcal{F} \circ \mathcal{H}. \tag{38}$$

By Proposition 1 *we get the following corollary.*

Corollary 3. $\mathcal{I} : C^\infty(\mathbb{R})^2 \longrightarrow \mathcal{M}$ *is a bijection.*

Now comes the main theorem.

Theorem 1. *The mapping* $\mathcal{I} : C^\infty(\mathbb{R})^2 \longrightarrow \mathcal{M}$ *defined by* (38) *is a homeomorphism.*

Before we give the proof, let us give a few lemmas.

We define some auxiliary maps and sets.

For $\beta \in \mathbb{N}^\times$:

$$\mathcal{R}(\beta) := \left\{ a \in \mathbb{N}^\beta : \sum_{j=1}^{\beta} j a_j = \beta \right\},$$

$$\mathcal{P}_\beta \; : \; \mathcal{R}(\beta) \ni a \longrightarrow \mathcal{P}_\beta(a) := \frac{\beta!}{\prod_{j=1}^{\beta} (j!)^{a_j} a_j!} \in \mathbb{N},$$

$$l_\beta \; : \; \mathcal{R}(\beta) \ni a \longrightarrow l_\beta(a) := \sum_{j=1}^{\beta} a_j \in \mathbb{N}.$$

For $(\lambda, \mu) \in \mathbb{N}^{\times} \times \mathbb{N}$:

$$\mathcal{R}(\lambda, \mu) := \left\{ a \in \mathbb{N}^{\lambda+1} : \sum_{j=1}^{\lambda} j a_j = \lambda, \sum_{j=0}^{\lambda} a_j = \mu \right\},$$

$$\mathcal{W}_{\lambda,\mu} : \mathcal{R}(\lambda, \mu) \ni a \longrightarrow \mathcal{W}_{\lambda,\mu}(a) := \frac{\mu!}{a_0} \frac{\lambda!}{\prod_{j=1}^{\lambda} (j!)^{a_j} a_j!} \in \mathbb{N},$$

$$c := (c^1, \ldots, c^{\beta}) \in X_{i=1}^{\beta} \mathcal{R}(\lambda_i, \mu_i),$$

$$\mathcal{T}(\lambda, \mu) := \left\{ a \in \mathbb{N}^{\lambda} : \sum_{j=1}^{\lambda} a_j = \mu \right\},$$

$$\mathcal{N}_{\lambda,\mu} : \mathcal{T}(\lambda, \mu) \ni a \longrightarrow \mathcal{N}_{\lambda,\mu}(a) := \frac{\mu!}{\prod_{j=1}^{\lambda} a_j!} \in \mathbb{N}.$$

Lemma 3. *Let $\mathbb{R} \supseteq \mathcal{O}_1$ and $\mathbb{R}^2 \supseteq \mathcal{O}_2$ be some open sets,*

$$h \in C^{\infty}(\mathcal{O}_1), \quad J \in C^{\infty}(\mathcal{O}_2) \quad \text{and} \quad \forall (t, x) \in \mathcal{O}_2 : 1 + J(t, x) > 0.$$

Then

1. $\forall \beta \in \mathbb{N}^{\times} : \partial^{\beta} \exp h = \left(\sum_{a \in \mathcal{R}(\beta)} \mathcal{P}_{\beta}(a) \prod_{j=1}^{\beta} (\partial^j h)^{a_j} \right) \exp h.$
2. $\forall \beta \in \mathbb{N}^{\times}\ \forall i \in \{1, 2\}$:

$$\partial_i^{\beta} \log(1 + J) = - \sum_{a \in \mathcal{R}(\beta)} (-1)^{l_{\beta}(a)} \frac{(l_{\beta}(a) - 1)!}{(1 + J)^{l_{\beta}(a)}} \mathcal{P}_{\beta}(a) \prod_{j=1}^{\beta} (\partial_i^j J)^{a_j}.$$

3. $\forall \beta, \gamma \in \mathbb{N}^{\times} : \partial_1^{\gamma} \partial_2^{\beta} \log(1 + J) =$

$$- \sum_{b \in \mathcal{R}(\gamma)} \sum_{a \in \mathcal{R}(\beta)} (-1)^{l_{\beta}(a)+l_{\gamma}(b)} \frac{(l_{\beta}(a) + l_{\gamma}(b) - 1)!}{(1 + J)^{l_{\beta}(a)+l_{\gamma}(b)}} \mathcal{P}_{\beta}(a) \mathcal{P}_{\gamma}(b)$$
$$\times \prod_{j=1}^{\beta} \prod_{k=1}^{\gamma} (\partial_1^j J)^{a_j} (\partial_2^k J)^{b_k} - \sum_{a \in \mathcal{R}(\beta)} \sum_{b \in \mathcal{T}(\beta,\gamma)} (-1)^{l_{\beta}(a)} \frac{(l_{\beta}(a) - 1)!}{(1 + J)^{l_{\beta}(a)}}$$
$$\times \mathcal{P}_{\beta}(a) \mathcal{N}_{\beta,\gamma}(b) \sum_{c \in X_{i=1}^{\beta} \mathcal{R}(b_i, a_i)} \prod_{k=1}^{\beta} \mathcal{W}_{b_k, a_k}(c^k) \prod_{j=1}^{b_k} (\partial_1^j \partial_2^k J)^{c_j^k}.$$

Remark 3. *One knows that*

$$\forall N \in \mathbb{N}\ \forall k \in \mathbb{N}^{\times} : \left| \left\{ x \in \mathbb{N}^k : \sum_{j=1}^{k} x_j = N \right\} \right| = \binom{N + k - 1}{k - 1}.$$

Thus

$$\forall\, \beta \in \mathbb{N}^{\times} :\mid \mathcal{R}(\beta) \mid \leq \sum_{j=1}^{\beta} \binom{\beta+k-1}{k-1} = \binom{2\beta}{\beta-1} < \infty.$$

Therefore, all sums in Lemma 3 are finite.

We skip a simple but lengthy proof of the Lemma 3.

Lemma 4. *Let (X, d_X), (Z, d_Z) be some metric spaces and let (Y, p_Y) be a semimetric space. If $X \supseteq K$ is compact and $Q : K \times Y \to Z$ is a continuous map, then*

$$\forall \epsilon > 0\, \forall y_0 \in Y\, \exists \delta > 0 :$$
$$(y \in K(y_0, \delta)) \Rightarrow (\forall\, x \in K : d_Z(Q(x, y), Q(x, y_0)) < \epsilon).$$

Proof of this lemma results from the proof of Maurin [3, Lemma IX.3.1].

Proof of Theorem 1.

Step 1. Let $(f_n)_{n=0}^{\infty}$ and $(g_n)_{n=0}^{\infty}$ be two sequences convergent in $C^{\infty}(\mathbb{R})$ to f and g, respectively. Then,

$$\forall (\mu, \nu) \in \mathbb{N}^{\times} \times \mathbb{N}\, \exists c_{\mu\nu} \in \mathbb{R}\, \forall n \in \mathbb{N} : p_{\mu\nu}(f_n) \leq c_{\mu\nu}.$$

Let us fix $c_{\mu\nu}$ for $(\mu, \nu) \in \mathbb{N}^{\times} \times \mathbb{N}$ (e.g., $c_{\mu\nu} := \inf\{\tilde{c}_{\mu\nu} \in \mathbb{R} : \forall n \in \mathbb{N}\; p_{\mu\nu}(f_n) \leq \tilde{c}_{\mu\nu}\}$) and denote

$$M_1 := \max\left\{ \binom{\beta}{\gamma} c_{\alpha\gamma} : \gamma \in \{0, 1, \ldots, \beta\} \right\},$$
$$M_2 := \max\left\{ \binom{\beta}{\gamma} p_{\alpha(\gamma-\beta)}(g) : \gamma \in \{0, 1, \ldots, \beta\} \right\},$$
$$M_{\alpha\beta} := \max\{M_1, M_2\}.$$

Then,

$$\begin{aligned} & p_{\alpha\beta}(f_n g_n - fg) \\ & \quad \leq \sum_{\gamma=0}^{\beta} \binom{\beta}{\gamma} p_{\alpha\gamma}(f_n) p_{\alpha(\beta-\gamma)}(g_n - g) \\ & \qquad + \sum_{\gamma=0}^{\beta} \binom{\beta}{\gamma} p_{\alpha\gamma}(f_n - f) p_{\alpha(\beta-\gamma)}(g) \\ & \quad \leq M_{\alpha\beta} \sum_{\gamma=0}^{\beta} (p_{\alpha(\beta-\gamma)}(g_n - g) + p_{\alpha\gamma}(f_n - f)) \xrightarrow[n\to\infty]{} 0, \end{aligned}$$

which means that the mapping

$$C^\infty(\mathbb{R})^2 \ni (f, g) \to fg \in C^\infty(\mathbb{R})$$

is continuous.

Define $h_n := f_n - f$ for $n \in \mathbb{N}$. We have $\mathrm{e}^f - \mathrm{e}^{f_n} = \mathrm{e}^f(1 - \mathrm{e}^{h_n})$, thus

$$p_{\alpha\beta}(\mathrm{e}^f - \mathrm{e}^{f_n}) \leq \sum_{\gamma=0}^{\beta} \binom{\beta}{\gamma} p_{\alpha\gamma}(\mathrm{e}^f) p_{\alpha(\beta-\gamma)}(1 - \mathrm{e}^{h_n}).$$

We get

$$p_{\alpha\beta}(\mathrm{e}^f - \mathrm{e}^{f_n}) \leq \tilde{M} \sum_{\gamma=0}^{\beta} p_{\alpha(\beta-\gamma)}(1 - \mathrm{e}^{h_n}),$$

where

$$\tilde{M} := \max\left\{ \binom{\beta}{\gamma} p_{\alpha\gamma}(\mathrm{e}^f) : \gamma \in \{0, 1, \ldots, \beta\} \right\} \in \mathbb{R}.$$

For $\gamma = \beta$ we have

$$p_{\alpha 0}(1 - \mathrm{e}^{h_n}) \leq \mathrm{e}^{p_{\alpha 0}(h_n)} - 1 \underset{n\to\infty}{\longrightarrow} 0.$$

For $\gamma > \beta \geq 0$ denote $\beta - \gamma := \rho + 1$ (so $\rho \geq 0$), then

$$p_{\alpha(\rho+1)}(1 - \mathrm{e}^{h_n}) \leq \sum_{\mu=0}^{\rho} \binom{\rho}{\mu} p_{\alpha(\mu+1)}(h_n) p_{\alpha(\rho-\mu)}(\mathrm{e}^{h_n}). \tag{39}$$

By Lemma 3 (see 1) we have

$$\exists M_0 \in \mathbb{R}\ \forall \mu \in \{0, 1, \ldots, \rho\}\ \forall n \in \mathbb{N}: \binom{\rho}{\mu} p_{\alpha(\rho-\mu)}(\mathrm{e}^{h_n}) < M_0,$$

therefore (39) gives

$$\forall (\alpha, \beta) \ni \mathbb{N}^\times \times \mathbb{N} : p_{\alpha\beta}(\mathrm{e}^f - \mathrm{e}^{f_n}) \underset{n\to\infty}{\longrightarrow} 0.$$

Since the map

$$\partial : C^\infty(\mathbb{R}) \ni f \longrightarrow \partial f \in C^\infty(\mathbb{R})$$

is continuous, the mapping $\mathcal{A}$ defined by (30) is continuous as the composition of continuous maps.

Step 2. Suppose $u \in C^\infty(\mathbb{R})$ and $\binom{a}{b} \in \mathbb{R}^2$. There is one and only one function $g \in C^\infty(\mathbb{R})$ such that

$$g'' = ug, \qquad g(0) = a, \qquad g'(0) = b. \tag{40}$$

Making substitution

$$\Psi : \mathbb{R} \ni s \longrightarrow \Psi(s) := \begin{pmatrix} g(s) \\ g'(s) \end{pmatrix} \in \mathbb{R}^2 \tag{41}$$

in (40) yields

$$\dot{\Psi} = \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix} \Psi, \qquad \Psi(0) = \begin{pmatrix} a \\ b \end{pmatrix}. \tag{42}$$

Let us denote the solution of (42) by $\Psi(\cdot; u)$ to indicate its dependence on $u \in C^\infty(\mathbb{R})$, and in addition let

$$A : C^\infty(\mathbb{R}) \ni u \longrightarrow A(u) := \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix} \in C^\infty(\mathbb{R}, M_{2\times 2}(\mathbb{R})),$$

$$B : C^\infty(\mathbb{R}) \ni h \longrightarrow B(h) := \begin{pmatrix} 0 & 0 \\ h & 0 \end{pmatrix} \in C^\infty(\mathbb{R}, M_{2\times 2}(\mathbb{R}).)$$

(In the sequel $A(u)(s) := A(u(s))$, $B(h)(s) := B(h(s))$.)

For $(\alpha, \beta) \in \mathbb{N}^\times \times \mathbb{N}$ and $\delta > 0$ we denote

$$s_{\alpha\beta}(\delta) := \{h \in C^\infty(\mathbb{R}) : \forall\gamma \in \{0, 1, \ldots, \beta\} p_{\alpha\gamma}(h) < \delta\}.$$

We notice that

$$\forall (u, h) \in C^\infty(\mathbb{R}) \times s_{\alpha 0}(1) : \sup_{t\in K_\alpha} \| A(u+h)(t) \| \le 1 + p_{\alpha 0}(u),$$

$$\forall h \in C^\infty(\mathbb{R}) : \sup_{t\in K_\alpha} \| B(h)(t) \| = p_{\alpha 0}(h).$$

Now, let us fix $u \in C^\infty(\mathbb{R})$ and $\alpha \in \mathbb{N}^\times$, and let us denote

$$M_u := 4 + p_{\alpha 0}(u).$$

We choose $\tau \in]0, 1/(1 + M_u)[$, $N(\tau) := \min\{n \in \mathbb{N}^\times : n\tau \ge \alpha\}$ and $K(\tau) := [-\tau N(\tau), \tau N(\tau)]$. Let $N \in \mathbb{N}^\times$ is such that $N \le 1/\tau \le N + 1$ and let $d := 1/(N + 2)$. Taking into account that

$$\forall t \in \mathbb{R} : \Psi(t; u) = \begin{pmatrix} a \\ b \end{pmatrix} + \int_0^t A(u(s))\Psi(s; u)\,\mathrm{d}s,$$

we get

$$\begin{aligned} \forall t \in [0, \tau] :& \| \Psi(t; u+h) - \Psi(t; u) \| \\ &\le | t | M_u \sup_{s\in[0,\tau]} \| \Psi(s; u+h) - \Psi(s; u) \| \\ &\quad + | t | \sup_{s\in K_\alpha} \| B(h(s))\Psi(s; u) \| . \end{aligned} \tag{43}$$

Let us fix $\epsilon > 0$. Applying Lemma 4 to the map

$$Q_u : \mathbb{R} \times C^\infty(\mathbb{R}) \ni (s, h) \longrightarrow Q_u(s, h) := B(h(s))\Psi(s; u) \in \mathbb{R},$$

and $(Y, p_Y) = (C^\infty(\mathbb{R}), p_{\alpha 0})$ gives

$$\exists \delta_0 > 0 \; \forall h \in s_{\alpha 0}(\delta_0) : \sup_{s \in K(\tau)} \| B(h(s))\Psi(s; u) \| < \epsilon d^{N(\tau)}.$$

Making use of this in (43) yields

$$\forall h \in s_{\alpha 0}(\delta_0) : \sup_{t \in [0,\tau]} \| \Psi(t; u + h) - \Psi(t; u) \| \\ \leq \tau M_u \sup_{s \in [0,\tau]} \| \Psi(s; u + h) - \Psi(s; u) \| + \epsilon d^{N(\tau)}.$$

Since τ is such that $0 < \tau < 1 - \tau M_u$ we get

$$\forall h \in s_{\alpha 0}(\delta_0) : \sup_{t \in [0,\tau]} \| \Psi(t; u + h) - \Psi(t; u) \| \leq \epsilon d^{N(\tau)}.$$

Applying previous considerations to the case $t \in [\tau, 2\tau]$ and making use of

$$\forall h \in s_{\alpha 0}(\delta_0) : \| \Psi(\tau; u + h) - \Psi(\tau; u) \| \leq \epsilon d^{N(\tau)},$$

one gets

$$\forall h \in s_{\alpha 0}(\delta_0) : \sup_{s \in [\tau, 2\tau]} \| \Psi(s; u + h) - \Psi(s; u) \| \leq \epsilon d^{N(\tau)-1}.$$

Repeated application of this procedure to $[2\tau, 3\tau], \ldots, [(N(\tau) - 1)\tau, N(\tau)\tau]$ gives

$$\forall h \in s_{\alpha 0}(\delta_0) \; \forall N \in \{1, 2, \ldots, N(\tau)\} : \\ \sup_{s \in [(N-1)\tau, N\tau]} \| \Psi(s; u + h) - \Psi(s; u) \| \leq \epsilon d^{N(\tau)-(N-1)}.$$

Similar reasoning applied to $[-\tau, 0], [-2\tau, -\tau], \ldots, [-N(\tau)\tau, -(N(\tau) - 1)\tau]$ enables to write

$$\forall h \in s_{\alpha 0}(\delta_0) : \sup_{s \in K(\tau)} \| \Psi(s; u + h) - \Psi(s; u) \| \leq \epsilon.$$

Since $K_\alpha \subseteq K(\tau)$, we get

$$\forall \epsilon > 0 \; \exists \delta_0 > 0 \; \forall h \in s_{\alpha 0}(\delta_0) : r_{\alpha 0}(\Psi(\cdot; u + h) - \Psi(\cdot; u)) \leq \epsilon. \tag{44}$$

Making use of Lemma 4, Eq. (44) and the estimate

$$\forall h \in s_{\alpha 0}(1) : \| (\partial \Psi)(t; u + h) - (\partial \Psi)(t; u) \| \\ \leq M_u \| \Psi(t; u + h) - \Psi(t; u) \| + \| B(h(t))\Psi(t; u) \|,$$

we get

$$\forall \epsilon > 0\, \exists \delta_1 > 0\, \forall h \in s_{\alpha 1}(\delta_1) : r_{\alpha 1}(\Psi(\cdot; u+h) - \Psi(\cdot; u)) \leq \epsilon.$$

Now, let us consider the identity

$$\begin{aligned}\forall \beta \in \mathbb{N}^{\times} &: \partial^{\beta}(\Psi(\cdot; u+h) - \Psi(\cdot; u)) \\ &= \sum_{\rho=0}^{\beta-1} \binom{\beta-1}{\rho} [A(\partial^{\rho}(u+h))\partial^{\beta-1-\rho}(\Psi(\cdot; u+h) - \Psi(\cdot; u))] \\ &\quad - \sum_{\rho=0}^{\beta-1} \binom{\beta-1}{\rho} B(\partial^{\rho} h)\partial^{\beta-1-\rho}\Psi(\cdot; u). \qquad (45)\end{aligned}$$

We notice that there are derivatives of order $0, 1, \ldots, \beta - 1$ in the right-hand side of (45). Using Lemma 4 for the map

$$Q_u^{\rho} : \mathbb{R} \times C^{\infty}(\mathbb{R}) \ni (s, h) \longrightarrow Q_u^{\rho}(s, h) := B(\partial^{\rho} h)\partial^{\beta-1-\rho}\Psi(s; u) \in \mathbb{R}^2,$$

where $\rho = 0, 1, \ldots, \beta - 1$ and where $(Y, p_Y) = (C^{\infty}(\mathbb{R}), p_{\alpha\rho})$, we get (by induction)

$$\forall \beta \in \mathbb{N}\, \forall \epsilon > 0\, \exists \delta_{\beta} > 0\, \forall h \in s_{\alpha\beta}(\delta_{\beta}) : r_{\alpha\beta}(\Psi(\cdot; u+h) - \Psi(\cdot; u)) \leq \epsilon.$$

Since our considerations apply to any $\alpha \in \mathbb{N}^{\times}$ and any $u \in C^{\infty}(\mathbb{R})$ we obtain that

$$C^{\infty}(\mathbb{R}) \ni u \longrightarrow \Psi(\cdot; u) \in C^{\infty}(\mathbb{R}, \mathbb{R}^2) \qquad (46)$$

(where $\Psi(\cdot; u)$ is a solution of (42)) is a continuous mapping. Taking into account that

$$\forall \Phi \in C^{\infty}(\mathbb{R}, \mathbb{R}^2) \quad \forall (\alpha, \beta) \in \mathbb{N}^{\times} \times \mathbb{N} : r_{\alpha\beta}(\Phi) \geq p_{\alpha\beta}(\Phi_1)$$

where

$$\Phi(t) =: \begin{pmatrix} \Phi_1(t) \\ \Phi_2(t) \end{pmatrix}$$

and solving (42) for

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

gives the conclusion that $\mathcal{B}$, defined by (31), is a continuous mapping.

Step 3. It is clear that both mappings $\mathcal{C}$ and $\mathcal{D}$ are continuous. The continuity of $\mathcal{E}$ can be proved by analogy to the case of mapping $\mathcal{A}$. The continuity of $\mathcal{G}$ results from the continuity of the mapping ($c \in \{-1, 1\}$)

$$\omega_c : C^{\infty}(\mathbb{R}) \ni f \longrightarrow \omega_c(f) \in C^{\infty}(\mathbb{R}^2),$$

where

$$\omega_c(f) : \mathbb{R}^2 \ni (t, x) \longrightarrow \omega_c(f)(t, x) := f(x + ct) \in \mathbb{R}.$$

The mapping $\mathcal{H}$ is continuous since it is a composition of continuous mappings.

Step 4. What is left is to prove that the mapping $\mathcal{F}$ is continuous. Suppose $(G_n)_{n=0}^{\infty}$ is a sequence of elements of $C_+^{\infty}(\mathbb{R}^2)$ convergent to $G \in C_+^{\infty}(\mathbb{R}^2)$.

Denote $H_n := G_n - G$. The sequence $(H_n)_{n=0}^{\infty}$ converges to zero in $C^{\infty}(\mathbb{R}^2)$. Since $\forall n \in \mathbb{N} : 1 + H_n/G > 0$, we have

$$\log G_n - \log G = \log \frac{G_n}{G} = \log\left(1 + \frac{H_n}{G}\right).$$

As $H_n \xrightarrow[n\to\infty]{} 0$ in $C^{\infty}(\mathbb{R}^2)$ we have

$$\forall \alpha \in \mathbb{N}^{\times}\ \exists N_\alpha \in \mathbb{R}\ \forall n > N_\alpha : q_{\alpha(0,0)}\left(\frac{H_n}{G}\right) < \frac{1}{2}.$$

Since $\forall x \in]-1/2, \infty[: | \log(1+x) | \leq 2 \mid x \mid$, we conclude that $\forall \alpha \in \mathbb{N}^{\times}\ \exists N_\alpha \in \mathbb{N}\ \forall n > N_\alpha$:

$$q_{\alpha(0,0)}\left(\log\left(1 + \frac{H_n}{G}\right)\right) \leq 2q_{\alpha(0,0)}\left(\frac{1}{G}\right) q_{\alpha(0,0)}(H_n).$$

Now, suppose $\mid \beta \mid = \beta_1 + \beta_2 > 0$ and denote $J_n := H_n/G$ for $n \in \mathbb{N}$. Since

$$\partial^{(\beta_1,\beta_2)} J_n = \sum_{\gamma_1=0}^{\beta_1} \sum_{\gamma_2=0}^{\beta_2} \binom{\beta_1}{\gamma_1}\binom{\beta_2}{\gamma_2}\left(\partial^{(\gamma_1,\gamma_2)}\frac{1}{G}\right)(\partial^{(\beta_1-\gamma_1,\beta_2-\gamma_2)} H_n)$$

we have

$$q_{\alpha(\beta_1,\beta_2)}(J_n) \leq M_{\alpha(\beta_1,\beta_2)} \sum_{\gamma_1=0}^{\beta_1}\sum_{\gamma_2=0}^{\beta_2} q_{\alpha(\gamma_1,\gamma_2)}(H_n),$$

where

$$M_{\alpha(\beta_1,\beta_2)} := \max\left\{\binom{\beta_1}{\gamma_1}\binom{\beta_2}{\gamma_2} q_{\alpha(\gamma_1,\gamma_2)}\left(\frac{1}{G}\right) : 0 \leq \gamma_1 \leq \beta_1, 0 \leq \gamma_2 \leq \beta_2\right\}.$$

Thus, we see that $J_n \xrightarrow[n\to\infty]{} 0$ in $C^{\infty}(\mathbb{R}^2)$. In particular we have

$$\forall \alpha \in \mathbb{N}^{\times}\ \exists N^{\alpha} \in \mathbb{N}\ \forall n > N^{\alpha} : \sup_{(t,x)\in K_\alpha^2} \left|\frac{1}{1 + J_n(t,x)}\right| \leq 2. \tag{47}$$

For $\beta \in \mathbb{N}^2$ let $D_\beta := \{(i,j) \in \mathbb{N}^2 : 1 \leq i + j \leq |\beta|\}$, $d_\beta := |D_\beta|$.
By (47) and Lemma 3 (see 2 and 3) we get that for $\beta \in \mathbb{N}^2$, $|\beta| \geq 1$ and $\alpha \in \mathbb{N}^{\times}$ there exists a polynomial $Q_{\alpha\beta} \in \mathbb{R}[x_1, \ldots, x_{d_\beta}]$ such that $Q_{\alpha\beta}(0) = 0$, $\deg Q_{\alpha\beta} \leq |\beta|$ and

$$\exists N^{\alpha} \in \mathbb{N}\, \forall n > N^{\alpha} : q_{\alpha(\beta_1,\beta_2)}(\log(1+J_n))$$
$$\leq Q_{\alpha\beta}(q_{\alpha(1,0)}(J_n), q_{\alpha(0,1)}(J_n), \ldots, q_{\alpha(|\beta|,0)}(J_n), q_{\alpha(0,|\beta|)}(J_n)).$$

Since $J_n \to 0$, it follows that

$$\forall(\alpha,\beta) \in \mathbb{N}^{\times} \times \mathbb{N}^2\, \forall \epsilon > 0\, \exists N_{\alpha\beta} \in \mathbb{N}\, \forall n > N_{\alpha\beta} : q_{\alpha\beta}(\log(1+J_n)) < \epsilon.$$

But $G \in C_+^{\infty}(\mathbb{R}^2)$ is an arbitrary function, therefore the mapping $\mathcal{F}$ is continuous. Finally, the mapping $\mathcal{I}$, defined by (38), is continuous as it is a composition of continuous mappings.

Step 5. It is clear that the mappings

$$\mathcal{K} : C^{\infty}(\mathbb{R}^2) \ni F \longrightarrow \mathcal{K}(F) := (F(0,\cdot), (\partial_t F)(0,\cdot)) \in C^{\infty}(\mathbb{R})^2$$

and

$$\mathcal{S} := \mathcal{K}_{|\mathcal{M}}$$

are continuous. Corollary 3 means that

$$\mathcal{I} \cdot \mathcal{S} = id_{\mathcal{M}} \quad \text{and} \quad \mathcal{S} \cdot \mathcal{I} = id_{C^{\infty}(\mathbb{R})^2}$$

This completes the proof.

4. Concluding remarks

The homeomorphism $\mathcal{I}^{-1} : \mathcal{M} \to C^{\infty}(\mathbb{R})^2$ gives $\mathcal{M}$ the structure of a topological manifold modeled on the Fréchet space $C^{\infty}(\mathbb{R})^2$.

We hope that making use of the ideas of Kriegl and Michor [4] one can define an infinite dimensional differential geometry on $\mathcal{M}$ and work out some geometrical methods useful not only in the study of the Liouville field theory but also, after generalization, in the examination of such important for physicists nonlinear field theories as the general relativity or the Yang–Mills theories [5].

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